The Tutte Polynomial and Applications

by

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Certificate of Approval

This is to certify that the accompanying thesis by Alexander M. Porter has been accepted in partial fulfillment of the requirements for graduation with Honors in Mathematics.

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Abstract

This paper gives a detailed overview of the Tutte polynomial and proves results that are simply stated in most of the current literature. It covers basic graph theory and matroid theory, basic properties of the Tutte polynomial, the recipe theorem and two applications of the recipe theorem: one to the Potts model and Ising model of statistical physics and one to the HOMFLY polynomial in knot theory.[1]

[1]MSC 2010: 05B35, 05C31, 05C83, 57M27, 82B20

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Introduction

W.T. Tutte was a British mathematician and codebreaker. During World War II, his work was instrumental in understanding the Lorentz cipher, a common German code. After the war, most of his work was done in the theories of graphs and matroids. One of his most notable accomplishments was the development of what he called the dichromate. We now refer to this as the Tutte polynomial.

Since its development, research in the Tutte polynomial has mostly been focused on applications and evaluations of the polynomial. As the purpose of this paper is to provide an introduction to the Tutte Polynomial and its applications that is accessible to those unfamiliar with the topic, we focus on two such applications: the Potts model of statistical physics and the HOMFLY polynomial in knot theory. In general, computing the Tutte polynomial is difficult. Research that is not focused on applications is generally focused on finding classes of matroids for which the polynomial is easy to compute and finding an easier method of computing the polynomial for all matroids. This area of research is beyond the scope of this paper and will not be covered. For information on this topic, see [4]. Most of the literature today skips over proofs of well established theorems. For the purpose of understanding, this paper proves these results.

This paper mirrors the development of the Tutte polynomial in that in both cases, invariants which we now know to be specializations are studied first and then generalized to the Tutte polynomial.

The Tutte polynomial is of two variables and is defined over all matroids. Because all graphs are matroids and graphs are easier to work with in most cases, the first half of section 1 of this paper is an introduction to graph theory and the chromatic polynomial which will provide a motivation for the development of the Tutte Polynomial. For a different motivation, see [5], W.T. Tutte's paper describing how he became acquainted with the polynomial. The second half of this section is devoted to matroid theory which will be used later to define the Tutte polynomial.

Section 2 defines the Tutte polynomial and one major theorem in the theory of the Tutte Polynomial, called the recipe theorem. This theorem plays a crucial role in studying applications of the Tutte polynomial as will be seen throughout sections 3 and 4. This section concludes by applying the recipe theorem to the chromatic polynomial.

Section 3 focuses on the application of the Tutte polynomial to the Potts model of statistical physics. It begins by describing the Potts model and why it is used in statistical physics. The section then demonstrates how to write one aspect of the Potts model in terms of the Tutte polynomial.

Section 4 describes the relationship between the polynomial and the HOMFLY polynomial in knot theory. It first describes the basic aspects of knot theory needed to understand the HOMFLY polynomial. It then uses the recipe theorem to write the HOMFLY polynomial in terms of the Tutte polynomial.

1 Preliminary Materials

In order to fully understand the Tutte polynomial, we must first consider some background information. This section begins with the relatively concrete and visual ideas of graph theory. We then generalize and abstract these concepts to discuss matroids s the Tutte polynomial is defined for all matroids.

1.1 Graph Theory

For those without experience in graph theory, this section serves as a brief introduction to the subject. For everyone, this section motivates the definition of the Tutte Polynomial. We begin first with the definition of a graph.

Definition 1 *A* ***graph*** *consists of a pair* (V, E)*, where* V *is the set of vertices and*

E is the set of edges. Each edge is an unordered pair of vertices.

Below is a depiction of a graph. The dots represent vertices. A line segment between vertices v_1 and v_2 represents the edge (v_1, v_2) or the edge (v_2, v_1) as these edges are equivalent under the definition of an edge.

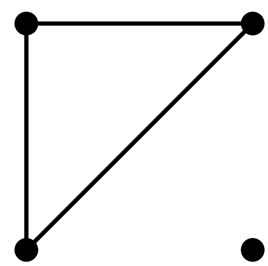

Figure 1: Example graph, G

As the definition of a graph is difficult to work with, we can think of the graph simply as the diagram that represents the graph. Some terms used throughout are included in the next few definitions.

Definition 2 *A pair of vertices are **adjacent** if they are connected by an edge.*

For example, in our example, the vertex in the upper left corner is adjacent to the vertex in the upper right corner but not to the vertex in the lower right corner.

Definition 3 *A graph is **connected** if between each pair of vertices, v and w, there is a set of vertices, $v = v_1, v_2, ..., v_n = w$ such that v_i and v_{i+1} are adjacent for all $i < n$.*

Visually, this means that you can travel by edges from one vertex to each other vertex, a fairly intuitive concept. Our example, G is not connected because the vertex in the bottom right corner is not adjacent to any other vertex. The diagram below depicts a graph that is connected and a graph that is not connected.

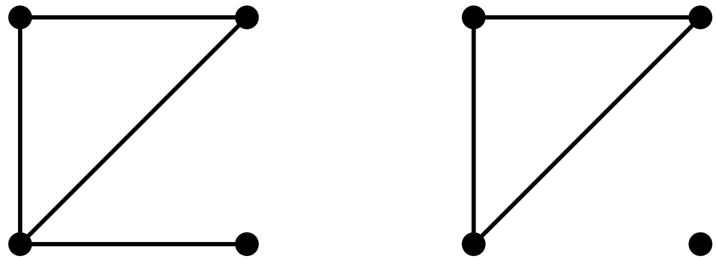

Figure 2: One connected graph and one disconnected graph

Using this same idea, we can define two new concepts.

Definition 4 *There is a* ***path*** *from v to w if there is a set of vertices, $v = v_1, v_2, ..., v_n = w$ such that v_i and v_{i+1} are adjacent for all $i < n$ and no vertex is used more than once.*

Definition 5 *A* ***cycle*** *is a path from a vertex to itself. That is a path such that $v_1 = v_n$.*

The picture below depicts a path between v and w in red and a cycle in green. Note that these are not the only cycle and path in this graph.

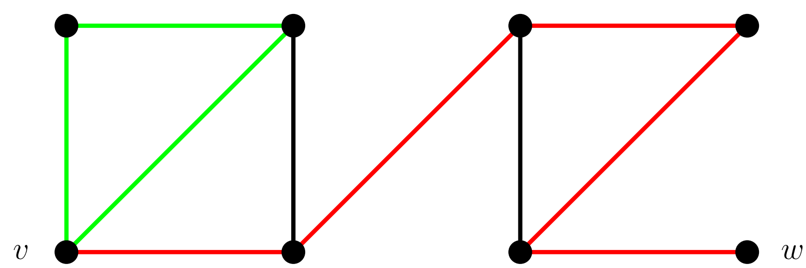

Figure 3: A path and a cycle

Suppose a graph is not connected. We can then discuss how many portions of the graph are connected. This idea is the motivation behind the following definition.

Definition 6 *A* ***connected component*** *of G is a maximal collection of vertices and edges such that for each pair of distinct vertices v and u in the collection, there* *is a path from v to u. The number of connected components of a graph is denoted $\kappa(G)$.*

Our example that is not connected has two connected components; the lower right vertex is its own connected component while the other three vertices and all three edges form the second. In addition, all connected graphs have one connected component. The next two definitions discuss types of edges.

Definition 7 *A **bridge** is an edge whose removal will cause the number of connected components to increase.*

Definition 8 *A **loop** is an edge whose endpoints are the same vertex.*

The graph below demonstrates two bridges with red edges and a loop with a purple edge. Note that the loop is a cycle on its own and the bridges are those edges that are not part of any cycle.

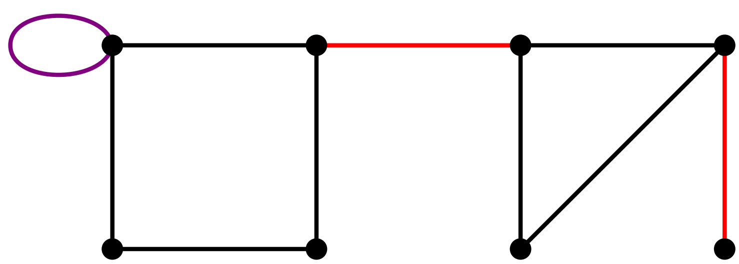

Figure 4: Two bridges and a loop

One particular graph that will be used throughout is the *single vertex graph* which is shown below its definition.

Definition 9 *The **single-vertex graph**, S, is the graph on one vertex and no edges.*

Figure 5: The single vertex graph, S

One of the many interesting topics in graph theory is graph colorings. This particular topic is chosen because of its relationship to the Tutte polynomial.

Definition 10 *A **proper coloring** of a graph is an assignment of a color to each vertex such that no two adjacent vertices are the same color.*

This leads to an interesting question; given k colors, how many different proper colorings are there of a graph? It turns out that the answer to this question is given by a polynomial in k.

Definition 11 *The **chromatic polynomial** of G, $P_G(k)$ is a polynomial defined for each graph whose output is the number of ways to properly color G with k colors.*

One of the most interesting properties of the chromatic polynomial is a recursive relationship which requires the taking of minors.

Definition 12 *A **minor** of a graph is the deletion or contraction of an edge, e. The **deletion** of e, denoted $G - e$, is the graph formed by removing e and leaving all other edges and vertices alone. The **contraction** of e, denoted $G \setminus e$, is formed by identifying the endpoints e into a single new vertex. That is replacing these vertices with one vertex with all edges except e left alone.*

The diagram below depicts a graph, named G, along with the deletion and contraction of the vertical edge on the left.

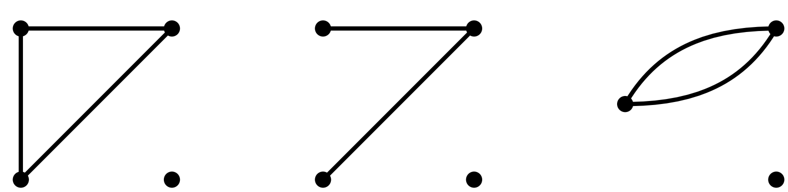

Figure 6: A graph and the deletion and contraction of an edge

The recursive relationship is then given by the following theorem. As this theorem is well established in graph theory, the proof is omitted.

Theorem 1 *For each graph, G, the following relationship holds:*

$$P_G(k) = P_{G-e}(k) - P_{G\backslash e}(k). \tag{1}$$

To demonstrate how this theorem is used in practice, it will be used to find the chromatic polynomial for an example graph. Since it can be shown that the chromatic polynomial of a graph with n vertices and 0 edges is k^n, we will reduce the graph until it has no edges. In addition, if a graph has a loop then its chromatic polynomial evaluates to 0. In each graph, the edge reduced over will be colored red. The polynomial below each bottom graph is the chromatic polynomial of that graph with the appropriate sign for its contribution to the chromatic polynomial of G.

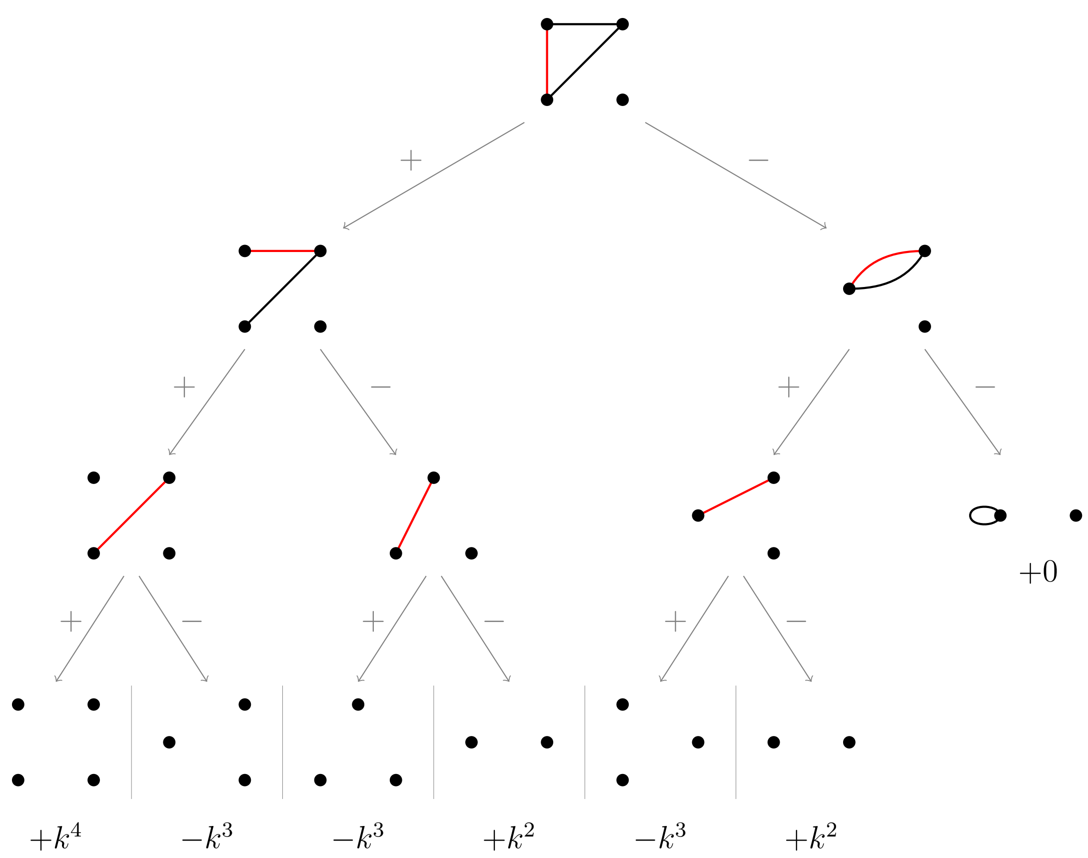

Figure 7: Finding the chromatic polynomial of G

This indicates that

$$P_G(k) = k^4 - 3k^3 + 2k^2. \tag{2}$$

This example will be used again in section 2.

1.2 Matroid Theory

Because the Tutte polynomial is defined over all matroids, we will cover the definition of a matroid and basic properties of matroids in order to better understand the Tutte Polynomial. For a deeper discussion of matroids, see [6].

Definition 13 *A* ***matroid****, $M = (E_M, r_M)$, is a mathematical structure that consists of a finite base set, E_M, and a rank function, $r_M(A)$, that maps subsets of E_M to non-negative integers and has the following properties:*

1. *$r_M(A) \le |A|$, for every subset A of E_M,*

2. *If $A \subseteq B$, then $r_M(A) \le r_M(B)$,*

3. *$r_M(A \cup B) + r_M(A \cap B) \le r_M(A) + r_M(B)$ for any $A, B \subseteq E_M$.*

The following theorem relates graphs to matroids.

Theorem 2 *Every graph, $G = (V_G, E)$, is a matroid where the base set is the edge set and the rank function is defined as follows:*

$$r_G(A) = |V_G| - \kappa(A) \tag{3}$$

where A is any subset of the edges of G and $\kappa(A)$ is the number of connected components of the graph (V_G, A).

Proof. Let us consider an arbitrary graph, G, with n vertices, edge set E and rank function r. First, it is clear that E is a finite set and thus fits the definition of the set of our matroid.

Condition 1: We will use induction on the number of edges in the chosen subset of E. Suppose our subset, A, has no edges. Then $r_G(A) = n - n = 0 \leq 0 = |A|$. Suppose $r_G(B) \leq |B|$ for all B with fewer than m edges. Let A have m edges. Choose any of the edges of A and call it e. Removing e leaves us with a subset, B, of E with $m-1$ edges. Thus $r_G(B) \leq m-1$. Now let us add e back. Adding this edge, does not change the number of vertices but it could decrease the number of connected components by 1. Thus $r_G(A) \leq r_G(B) + 1 \leq m - 1 + 1 = m$. Thus for each subset of E, part 1 of the rank function definition holds.

Condition 2: Let us consider sets A and B such that $A \subseteq B \subseteq E$. Suppose $\kappa(A) = m$. Then since B contains all of the edges of A, we see that $\kappa(B) \leq m$. Therefore $r(A) = n - m$ and $r(B) \geq n - m$. Therefore $r(A) \leq r(B)$ and the second condition on the rank function holds.

Condition 3: Suppose A and B are subsets of E. We will induct on the number of edges in A or B but not both. In symbols, this is $|\,(A \cup B) \setminus (A \cap B)\,| = m$. Consider the case where $m = 0$. Then $A = B = A \cup B = A \cap B$. Therefore $r(A) = r(B) = r(A \cup B) = r(A \cap B)$. Therefore it is clear that $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$. Suppose this is true for $0, 1, 2, ..., m-1$ and that m is non-zero. Since m is non-zero, without loss of generality, there is an edge, e, in $A \setminus B$. Consider the sets $A \setminus \{e\}$ and B. Since $|\,((A \setminus \{e\}) \cup B) \setminus ((A \setminus \{e\}) \cap B)\,| = m - 1$, it follows from the induction hypothesis that $r([A \setminus \{e\}] \cup B) + r([A \setminus \{e\}] \cap B) \leq r(A \setminus \{e\}) + r(B)$. Since e is not in B, it is clear that $r(A \cap B) = r([A \setminus \{e\}] \cap B)$ and that $r(B) = r(B)$. Suppose that $r([A \setminus \{e\}] \cup B) = r(A \cup B) - 1$. Then we know that $A \setminus \{e\}$ has one more connected component than A. Therefore $r(A \setminus \{e\}) = r(A) - 1$. Therefore we see that $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$.

From these three facts, we see that (E_G, r) is a matroid for any graph. ■

Definition 14 *A subset, A, of E is* ***independent*** *if $r(A) = |A|$. If a subset is not independent, it is* ***dependent****. We will say $A \in \mathcal{I}$ if A is independent.*

Although a matriod is defined by its rank function, it could be just as well defined by its independent sets. Such a definition is as follows.

Definition 15 *A **matroid** $M = (E, \mathcal{I})$ is a finite set E and a set $\mathcal{I}$ of subsets of E such that the following are true:*

1. *$\emptyset \in \mathcal{I}$ (equivalently $\mathcal{I}$ is non-empty),*
2. *If $A \subseteq B$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$,*
3. *If $A, B \in \mathcal{I}$ and $|A| = |B| + 1$ then there is an $x \in A \setminus B$ such that $B \cup \{x\} \in \mathcal{I}$.*

Using this definition of a matroid, we can define the rank function of A, a subset of E, as the size of the largest independent set contained in A. As the purpose of this paper is to study the Tutte polynomial, a proof that the two definitions of a matroid are equivalent is omitted.

The following theorem gives an easy way to identify the independent sets in a graph.

Theorem 3 *A subset of the edges of a graph is independent if and only if it contains no cycle.*

Proof. In this proof $A \setminus \{e\}$ indicates set subtraction, not contraction on e.

Suppose A is a subset of the edges of G and contains a cycle. Choose an edge of this cycle and call it e. We see that $\kappa(A) = \kappa(A \setminus \{e\})$ and thus $r(A) = r(A \setminus \{e\})$. Since $r(A) = r(A \setminus \{e\}) \leq |A \setminus \{e\}| = |A| - 1$, we see that $r(A) < |A|$ and A is not independent.

Suppose B is a subset of the edges of G and contains no cycle. We will induct on the number of edges in B. If $|B| = 0$ then $\kappa(B) = |V_G|$ and $r(B) = 0 = |B|$. Suppose that all subsets of E_G with fewer than $|B|$ edges and no cycles are independent. Since B has no cycles, neither does $B \setminus \{e\}$ where e is an arbitrary edge of B. Thus

$r(B \setminus \{e\}) = |B \setminus \{e\}|$. Note that the removal of e increases the number of connected components. From this, we see that $r(B) = r(B \setminus \{e\}) + 1$. Thus $r(B) = |B|$ and B is independent. ■

The concept of independence leads to the following definition of the direct sum of matroids.

Definition 16 *The **direct sum** of two matroids, denoted $M_1 \oplus M_2$, is the matroid generated by $E = E_{M_1} \cup E_{M_2}$ with $\mathcal{I} = \{A \cup B : A \in \mathcal{I}_{M_1}, B \in \mathcal{I}_{M_2}\}$.*

That is, the direct sum is the matroid created by taking all the elements of each set and stating that a set in $M_1 \oplus M_2$ is independent if it is the union of an independent set of M_1 and an independent set in M_2.

Pictorially, this definition is quite simple for graphs. The direct sum is generated by placing G_1 and G_2 side by side and defining the rank function exactly as it was defined in section 2.1. For example, if G is the example graph from above, then $G \oplus G$ is shown below.

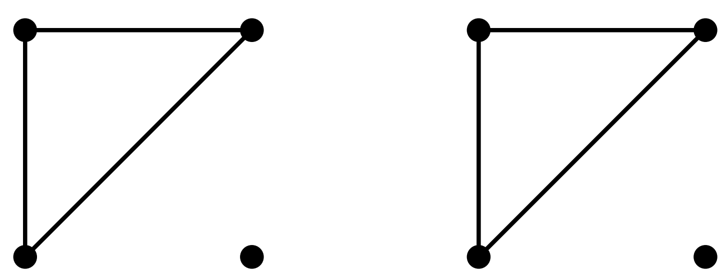

Figure 8: The direct sum $G \oplus G$

From theorem 3, we can see that the largest independent set in G has two elements and thus the largest independent set in $G \oplus G$ has four elements.

As loops and bridges have been defined for graphs, the rank function can be used to define these same terms more generally for matroids.

Definition 17 *In a matroid, a **loop** is an element, e, such that $r(e) = 0$. A **bridge** is an element, a, such that $r(A \cup \{a\}) = r(A) + 1$ for any independent set, A.*

From this definition, we can see that any set containing a loop is dependent and if A is an independent set, so is $A \cup a$ if a is a bridge.

Note that since graph loops are also cycles, they are dependent by theorem 3 and thus are matroid loops. Also, since graph bridges are not part of any cycle, adding one to any independent set will not create a new cycle. Thus graph bridges are also matroid bridges.

The last thing to cover about matroids is the taking of minors. As noted in section 1.1, a minor of a graph is either the deletion or contraction of an edge. Below is a similar definition for the taking of minors of matroids.

Definition 18 *The **restriction** to S of a matroid (E, r), is the matroid (S, r_S) where r_S is the rank function of E restricted to subsets of S. The restriction to S is denoted $M|S$. The **deletion** of an element, e, of E is the restriction to $E \setminus \{e\}$ and is denoted $M - e$.*

From this definition, note that the deletion of an edge, e, from a graph, which we have denoted $G - e$, is the restriction of the edges of the graph to $E \setminus \{e\}$.

Definition 19 *Given a matroid $M = (E, r)$, the **contraction** of a subset, T, or E is the matroid $M \setminus T = (E \setminus T, r')$ where $r'(A) = r(A \cup T) - r(T)$ for any subset, A, of $E \setminus T$.*

Note that the contraction of an edge, e, in a graph, which we have denoted $G \setminus e$, is then $M \setminus \{e\}$, the contraction on $\{e\}$.

2 The Tutte Polynomial

The Tutte Polynomial is a generalization of the chromatic polynomial, as well as many other interesting invariants of graphs, that retains a similar recursive property to that of the chromatic polynomial described in section 1. It is a polynomial in two variables that is defined over all matroids. Although this paper focuses on properties of the Tutte Polynomial as it relates to graphs, many theorems about the Tutte Polynomial are stated and proven in terms of matroids. This style will be replicated here. This section defines the Tutte Polynomial and provides one of the most important theorems in the theory of the Tutte Polynomial, the recipe theorem. For more information on the Tutte polynomial including a wide variety of applications, see [7].

2.1 The Definition

Using the properties of matroids described in section 1 and the fact that all graphs are matroids, the Tutte Polynomial and theorems about it can be introduced in terms of matroids with the knowledge that all will apply to graphs. To begin, the Tutte Polynomial will be defined.

Definition 20 *The **Tutte Polynomial**, $T(M;x,y)$ is a polynomial of two variables that is defined for every matroid as follows:*

1. *If E_M is empty, then $T(M;x,y) = 1$,*
2. $T(M;x,y) = yT(M \setminus e;x,y)$ *if e is a loop,*
3. $T(M;x,y) = xT(M - e;x,y)$ *if e is a bridge,*
4. $T(M;x,y) = T(M - e;x,y) + T(M \setminus e;x,y)$ *for any other element, e.*

As with the chromatic polynomial, we can use these relationships with the Tutte Polynomial to reduce a matroid. Since graphs are much easier to viualize than a general matroid, we will use a graph to illustrate this reduction. Again, we will be reducing over the red edge in each case and the polynomial below each bottom graph indicates the contribution of that graph to the Tutte Polynomial of G.

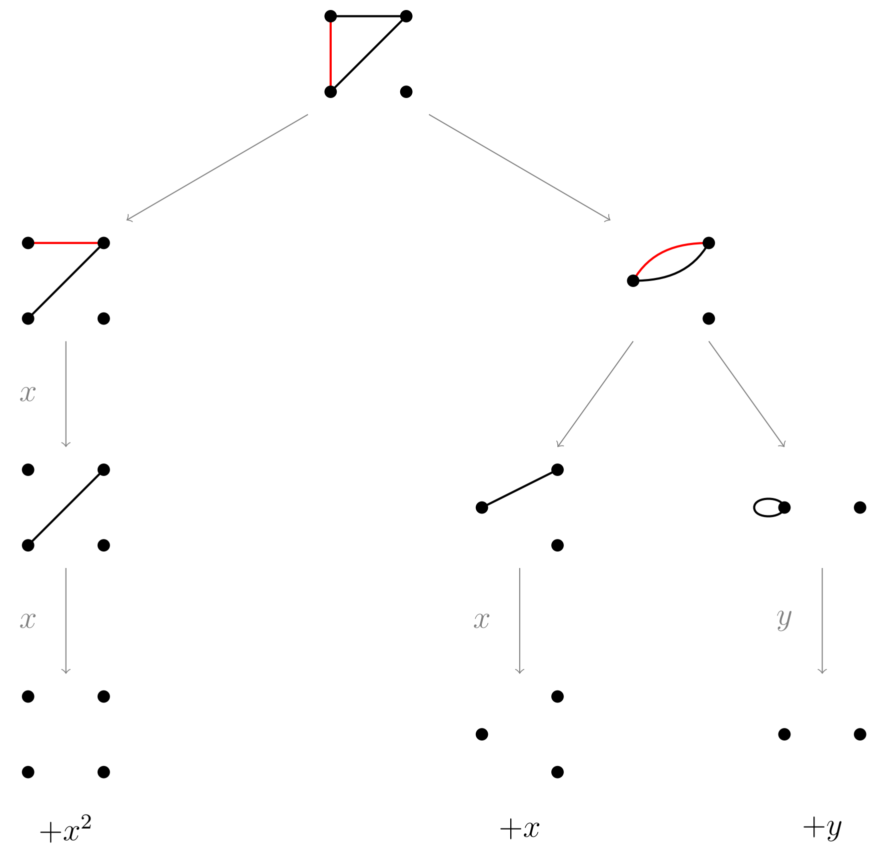

Figure 9: Finding the Tutte polynomial of G

This yields that the Tutte polynomial for our example graph is given by

$$T(G; x, y) = x^2 + x + y. \tag{4}$$

One should take note of the similarities and differences between the decomposition above and the chromatic decomposition presented in section 1. The most notable difference is how bridges and loops are treated. As bridges and loops are decomposed differently, the reduction for the Tutte Polynomial is much simpler. For example, the Tutte Polynomial for the following graph is easily computed.

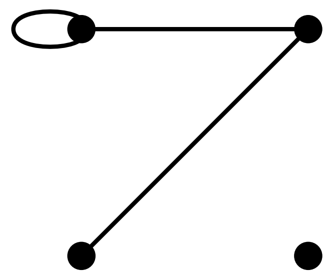

Figure 10: A graph for which the Tutte polynomial is easily computed

Since this this graph has only bridges and loops, the Tutte Polynomial for it is x raised to the number of bridges times y raised to the number of loops, x^2y. This means that our decomposition on the left branch could have been stopped after one iteration and the right branch could have been stopped after two iterations.

It is also important to note that the Tutte polynomial is unique; that is, for any matroid there is exactly one Tutte polynomial. This fact is evident from the defintion of the Tutte polynomial in section 15.4 of [6]. The converse is not true. It is possible to have two matroids with the same Tutte polynomial.

2.2 The Recipe Theorem

The following theorem demonstrates the relationship between the Tutte Polynomial and the chromatic polynomial as well as many other applications. A simple explanation of the theorem is that if you have an invariant that has a recursive relation similar enough to that of the Tutte Polynomial, the invariant is essentially an evaluation of the Tutte Polynomial.

Theorem 4 (The Recipe Theorem) *Suppose $\mathcal{C}$ is a class of matroids which is closed under direct sums and the taking of minors. If a function, f, is well defined over $\mathcal{C}$ and satisfies*

1. $f(M) = af(M - e) + bf(M \setminus e)$ *if e is neither a bridge nor a loop,*

2. $f(M) = x_0 f(M-e)$ *if* e *is a bridge,*

3. $f(M) = y_0 f(M \setminus e)$ *if* e *is a loop,*

and evaluates to 1 *on the empty set, then* f *is given by*

$$f(M) = a^{|E|-r(E)} b^{r(E)} T\left(M; \frac{x_0}{b}, \frac{y_0}{a}\right) \tag{5}$$

where $M = (E, r)$ *is any matroid in* $\mathcal{C}$.

Proof. Suppose f is a function that satisfies equation 5. We will prove that f satisfies the three conditions of the recipe theorem.

First note that if E is empty, then $f(M) = a^0 b^0 T\left(M; \frac{x_0}{b}, \frac{y_0}{a}\right) = 1$.

Condition 1*:* Suppose we are reducing over an element that is neither an bridge nor a loop. Since we know about properties of the Tutte polynomial's reduction over elements that are neither bridges nor loops, we can use the same reduction on equation 5.

$$f(M) = a^{|E|-r(E)} b^{r(E)} \left[T\left(M-e; \frac{x_0}{b}, \frac{y_0}{a}\right) + T\left(M \setminus e; \frac{x_0}{b}, \frac{y_0}{a}\right)\right]$$

Noting that removing an element that is neither a loop nor a bridge reduces $|E|$ by 1 but leaves the rank of E the same. Contracting on an element that is neither a loop nor a bridge reduces both $|E|$ and the rank of E by 1. This fact along with elementary algebra leaves us with condition 1.

Condition 2*:* Suppose we are reducing over a bridge. Since we know about the reduction of the Tutte polynomial over a bridge, we see that equation 5 becomes

$$f(M) = a^{|E|-r(E)} b^{r(E)} \frac{x_0}{b} T\left(M-e; \frac{x_0}{b}, \frac{y_0}{a}\right).$$

Since removal of a bridge reduces both $|E|$ and the rank of E by 1, it becomes clear that f satisfies condition 2.

Condition 3*:* Suppose we are reducing over a loop. Reduction of the Tutte polynomial yields

$$f(M) = a^{|E|-r(E)} b^{r(E)} \frac{y_0}{a} T\left(M \setminus e; \frac{x_0}{b}, \frac{y_0}{a}\right).$$

As with the others, since removing a loop reduces $|E|$ by 1 and leaves the rank of E unchanged, we can see that this reduces to show that f satisfies condition 3.

Since the Tutte polynomial of a matroid is unique, the result then follows. ■

Although the recipe theorem gives a way to write many functions of matroids in terms of the Tutte polynomial, graphs are a fairly restricted class of matroids and thus need fewer conditions in order to make the recipe theorem work. The next two lemmas are intended to demonstrate some properties of graphs in order to make the recipe theorem easier to use. The fist states that graphs are a class of matroids that are closed under the taking of minors and direct sums. The second makes the condition on how the function behaves over direct sums easier to manage.

Lemma 1 *Graphs are closed under the taking of minors and under direct sums.*

Proof. As can be seen by the definitions of deletions, contractions and direct sums for graphs, each of these operations, when performed on a graph (or two for direct sums), produces another graph. ■

Lemma 2 *If f is a function defined for all graphs and $f(G_1 \oplus G_2) = f(G_1)f(G_2)$ whenever G_1 and G_2 have no edges and f satisfies the following:*

1. $f(G) = af(G - e) + bf(G \setminus e)$ *if e is neither a bridge nor a loop,*

2. $f(G) = x_0 f(G - e)$ *if e is a bridge,*

3. $f(G) = y_0 f(G \setminus e)$ *if e is a loop,*

then $f(G_1 \oplus G_2) = f(G_1)f(G_2)$ *for any graphs,* G_1 *and* G_2*.*

Proof. The first condition of this theorem give us the base case for a proof in which we induct on the number of edges in $G_1 \oplus G_2$. Suppose $n \geq 1$ and $f(G_1 \oplus G_2) = f(G_1)f(G_2)$ whenever $G_1 \oplus G_2$ has fewer than n edges. Suppose further that $G \oplus H$ has n edges. Let e be one of these edges. Without loss of generality, e is an element of G. We now consider three cases:

Case 1: Suppose e is neither a bridge nor a loop. By condition 1, we know that $f(G \oplus H) = af((G-e) \oplus H) + bf((G \setminus e) \oplus H)$. Since both $(G-e) \oplus H$ and $(G \setminus e) \oplus H$ have $n-1$ edges, our hypothesis implies that $f((G-e) \oplus H) = f(G-e)f(H)$ and $f((G \setminus e) \oplus H) = f(G \setminus e)f(H)$. It then follows that $f(G \oplus H) = (af(G-e) + bf(G \setminus e))\, f(H)$. Since e is a neither a loop nor a bridge, we see that $f(G \oplus H) = f(G)f(H)$.

Case 2: Suppose e is a bridge. By condition 2, $f(G \oplus H) = x_0 f((G-e) \oplus H)$. Since $(G-e) \oplus H$ has $n-1$ edges, our hypothesis implies that $f((G-e) \oplus H) = f(G-e)f(H)$. We then see that $f(G \oplus H) = x_0 f(G-e)f(H)$. Since e is a bridge. It is now clear that $f(G \oplus H) = f(G)f(H)$.

Case 3: Suppose e is a loop. By condition 3, $f(G \oplus H) = y_0 f((G \setminus e) \oplus H)$. Since $(G \setminus e) \oplus H$ has $n-1$ edges, our hypothesis implies that $f((G \setminus e) \oplus H) = f(G \setminus e)f(H)$. We then see that $f(G \oplus H) = x_0 f(G \setminus e)f(H)$. Since e is a loop. It is now clear that $f(G \oplus H) = f(G)f(H)$.

Therefore $f(G \oplus H) = f(G)f(H)$ when our conditions are met. ■

These two lemmas lead us to our next theorem. The purpose of this theorem is to adjust the conditions on the recipe theorem to make it easier to use when the class of matroids is those matroids that are also graphs.

Theorem 5 (The Recipe Theorem for Graphs) *If a function, f, is well defined for all graphs an satisfies*

1. $f(G) = af(G-e) + bf(G \setminus e)$ *if e is neither a bridge nor a loop,*

2. $f(G) = x_0 f(G-e)$ *if e is a bridge,*

3. $f(G) = y_0 f(G \setminus e)$ *if e is a loop,*

for all connected graphs and satisfies $f(G) = 1$ if G has no edges, then f is given by

$$f(G) = a^{|E|-r(E)} b^{r(E)} T\left(G; \frac{x_0}{b}, \frac{y_0}{a}\right). \tag{6}$$

Proof. We will prove that functions that satisfy these properties on graphs satisfy all of the conditions of the recipe theorem when the class of matroids is graphs. The result will then follow. By lemma 1, graphs are a class of matroids that are closed under direct sums and the taking of minors. Suppose f is a function that satisfies the conditions of this theorem. Then by lemma 2, f satisfies $f(G_1 \oplus G_2) = f(G_1)f(G_2)$ for any graphs G_1 and G_2. The only remaining conditions of the recipe theorem are listed as conditions 1 through 4 in this theorem. Therefore, by the recipe theorem, f is given by equation 6. ■

From this, we can see that graphs are a particularly nice class of matroids that make working with the recipe theorem much easier. There is one more particularly nice property of graphs that will be utilized in later sections.

Theorem 6 *For any graph G, $T(G-e; x, y) = T(G \setminus e; x, y)$ if e is a bridge or a loop.*

Proof. Suppose e is a bridge. Note that $G - e = G_1 \oplus G_2$ for some G_1 and G_2. Therefore $T(G-e; x, y) = T(G_1; x, y)T(G_2; x, y)$. Next note that $G \setminus e$ consists of G_1 and G_2 adjoined at one vertex. We will call this $G_1 \times G_2$. Thus $G \setminus e = G_1 \times G_2$. Reduce $G_1 \times G_2$ by each edge of G_1. We now have a tree similar to the one in figure 9 but unfinished. Note that each branch has the entirety of G_2 still. Now when we reduce over each edge of G_2, we will have a common factor of $T(G_2; x, y)$ at the bottom of each branch. By factoring this out we see that $T(G_1 \times G_2; x, y) = T(G_1; x, y)T(G_2; x, y)$. Thus $T(G - e; x, y) = T(G \setminus e; x, y)$.

Suppose e is a loop. Since the endpoints of e are the same vertex, we see that the deletion of e and the contraction of e are in fact the same graph and thus have the same Tutte polynomial. ■

This theorem allows us to switch between $G - e$ and $G \setminus e$ in conditions 2 and 3 of the recipe theorem and the recipe theorem for graphs.

2.3 The Chromatic Polynomial as a Specialization

Using the recipe theorem, let us examine the known recurrence relationship for the chromatic polynomial. Recall that the recipe theorem requires that $f(G) = 1$ if G has no edges. Unfortunately, $P_G(k) = k^n$ if G has no edges and n vertices.

To fix this problem, let us define a new function. This new polynomial will look much like the chromatic polynomial but with one major distinction. We will define this function as

$$p_G(k) = P_G(k)/k^{\kappa(G)}. \tag{7}$$

It is this new polynomial to which we can apply the recipe theorem.

Theorem 7 *The chromatic polynomial is given by the following equation*

$$P_G(k) = k^{\kappa(G)}(-1)^{r(E)}T\left(G; 1-k, 0\right). \tag{8}$$

Proof. To prove this theorem, we will apply the recipe theorem for graphs to our new polynomial, p_G. First note that by the definition of p_G, it evaluates to 1 whenever G has no edges.

Condition 1: To prove that P_G satisfies condition 1 of the recipe theorem for graphs, recall that

$$P_G(k) = P_{G-e}(k) - P_{G\setminus e}(k).$$

From this, we see that the following is valid for any edge e that is neither a bridge nor a loop

$$p_G(k) = p_{G-e}(k) - p_{G\setminus e}(k)$$

because removing or contracting upon such an edge does not change the number of connected components of the graph. This means that p_G satisfies the first condition of the recipe theorem for graphs. In the language of the recipe theorem, this demonstrates that $a = 1$ and $b = -1$.

Condition 2: To prove that P_G satisfies condition 2 of the recipe theorem for graphs, again recall the recursive relationship for P_G. If we instead consider a bridge, removal of e increases the number of connected components by 1. This demonstrates that

$$p_G(k) = kp_{G-e}(k) - p_{G\setminus e}(k).$$

By theorem 6, we know that $P_{G\setminus e}(k) = P_{G-e}(k)$ and thus $p_{G\setminus e}(k) = p_{G-e}(k)$. This simple substitution gives us the following equation

$$p_G(k) = (k-1)p_{G-e}(k).$$

This indicates that p_G satisfies the second condition of the recipe theorem with $x_0 = k - 1$.

Condition 3: To prove that P_G satisfies condition 3 of the recipe theorem for graphs, consider the reduction of a loop in the recursive relationship of P_G. Since deletion or contraction on a loop does not change the number of connected components, we see that

$$p_G(k) = p_{G-e}(k) - p_{G\setminus e}(k).$$

By theorem 6, this equation reduces to

$$p_G(k) = p_{G\setminus e}(k) - p_{G\setminus e}(k) = 0.$$

This demonstrates that p_G satisfies the third condition of the recipe theorem and that $y_0 = 0$.

The recipe theorem for graphs combines the results for a, b, x_0 and y_0 and gives the following equation

$$p_G(k) = (-1)^{r(E)} T\left(G; 1-k, 0\right). \tag{9}$$

Since $p_G(k) = P_G(k)/k^{\kappa(G)}$, equation 8 is valid. ■

With this theorem, there is another way of calculating the chromatic polynomial of a graph. To demonstrate, consider our example graph from section 1. In equation 2 we calculated the chromatic polynomial of our example graph to be $P_G(k) = k^4 - 3k^3 + 2k^2$. In equation 4 we found the Tutte polynomial of the same graph to be $T(G; x, y) = x^2 + x + y$. Combining this with equation 8 from the above theorem yields $P_G(k) = k^4 - 3k^3 + 2k^2$ as expected.

2.4 Other Specializations

Since the chromatic polynomial is a specialization of the Tutte polynomial, one may wonder if there are any other properties of graphs that are specializations.

One such example is the *complexity*, the number of *spanning trees*, of a connected graph. A spanning tree of a graph is a minimal subset of the edges of the graph that allows the vertices of the graph to still be connected. Below are two spanning trees of a graph. It is interesting to note that in this particular graph, both the red set of edges and the blue set of edges form a spanning tree. This is not always the case.

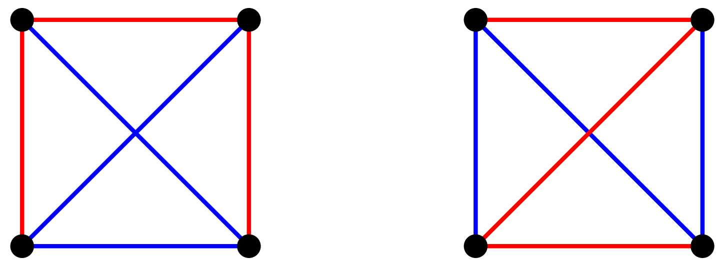

Figure 11: Spanning trees of a graph, K_4

It is clear that using only the red edges, there is a path from any vertex to any other vertex. In addition, removing one red edge will remove a path between two vertices. Thus each of these sets of edges is a spanning tree for the graph. Without going into detail, using the notation of the recipe theorem, $a = b = x_0 = y_0 = 1$. Therefore at $T(G; 1, 1)$ gives the number of spanning trees of G. Since $T(K_4; x, y) = x^3 + 3x^2 + 2x + 4xy + 2y + 3y^2 + y^3$, there are 16 different spanning trees of K_4.

Two other specializations of the Tutte polynomial will be covered in more detail. Section 3 covers the Potts model of statistical physics and section 4 covers the HOMFLY polynomial in knot theory.

3 Statistical Physics

One application of the Tutte Polynomial is with the Ising and Potts models of statistical physics. Since the Ising model is a specific case of the Potts model, it is easiest start with the Ising model. A generalization to the Potts model is then much simpler. In this section, we will first describe the Ising and Potts models in terms of elementary statistical physics. These models are of particular interest to mathematicians because they are exactly solvable while many models of physics are not. We will then use the recipe theorem as described in section 2.2 to describe these models with the Tutte Polynomial. For more information on the relationship between the Potts model and the Tutte polynomial, see [8].

3.1 The Potts and Ising Models

The purpose of the Ising model is to describe a 2-state system, that is a system in which a group of objects can be in one of two states which we will call $+1$ and -1. A basic example of such a system is the lights in a building. Each light can be in one of two states: on $(+1)$ or off (-1). This particular example is a little too simple because the state of one light is completely independent from the state of the other lights.

In addition, we use the state of the system to describe the state of every particle at once. For example, the building is in state β if all of the lights on the first floor are on and all of the lights on every other floor are off. We will denote an arbitrary system-state by σ.

The Potts model is a generalization of the Ising model in which more than two states are allowed. In general, we allow Q states for each object in the system. Thus, the Ising model is the 2-state Potts model. From here on, we will discuss only the Potts model with the knowledge that the Ising model is a specific case of the Potts model.

In statistical physics, the Potts model is foremost used to describe the states of electrons in a material. In this setting, each electron has a spin (state) associated with it that causes the electron to act like a tiny magnet pointing in one of Q directions.

The Potts model is used to represent a system in which the objects are in an array. To relate this back to graph theory, we will build a graph to represent this array. Consider an array with N particles. Each particle will be represented with vertex in our graph. Each of these vertices has an associated spin, which will be represented by a color attributed to the vertex in our graph.

The Potts model states that the energy between two nearest-neighbor particles is zero if the particles have the same spin and equal to a constant, called J, if the spins are different. Thus, nearest-neighbor vertices will be called adjacent and connected by and edge. This means that the energy of the system in state σ is given by

$$E(\sigma) = J \sum_{i \sim j} (1 - \delta(\sigma_i, \sigma_j)) \tag{10}$$

where the sum is over all pairs of adjacent vertices, σ_i is the spin at vertex i in state σ and δ is the Kronecker delta function. Below is a picture describing the 2-state model for an array with $N = 9$ particles.

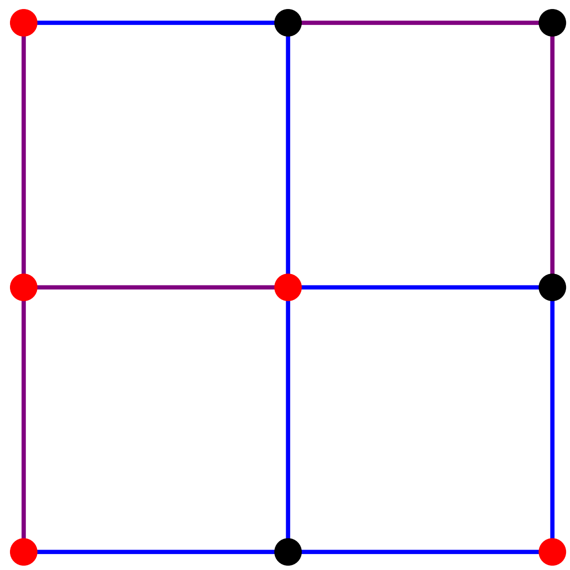

Figure 12: An example state

Note that there are 5 pairs of adjacent vertices with the same spin (represented by purple edges) and 7 with different spins (represented by blue edges). Thus, by equation 10, we see that $E = J(5(1-1) + 7(1-0)) = 7J$.

From properties of the physical system, it has been determined that the probability of being in state σ is

$$Pr(\sigma) = e^{-\beta E(\sigma)}/Z \tag{11}$$

where $\beta = 1/k_B T$ where k_B is the Boltzmann constant, T is the temperature and Z is the *partition function* and is independent of the state of the system.

Since our goal is to find the probability of being in state σ, the only thing we need to find is the partition function. Unfortunately this is not as easy as it may sound.

In order to find the partition function, note that the sum of the probabilities must be 1. From this we see that

$$1 = \sum_{\sigma} \frac{1}{Z} e^{-\beta E(\sigma)}.$$

Replacing E with equation 10 yields

$$1 = \sum_{\sigma} \frac{1}{Z} \exp\left(-\beta J \sum_{i \sim j} (1 - \delta(\sigma_i, \sigma_j))\right).$$

Solving for Z then results in

$$Z(G; Q, \beta J) = \sum_{\sigma} \exp\left(-\beta J \sum_{i \sim j} (1 - \delta(\sigma_i, \sigma_j))\right) \tag{12}$$

where we have noted that Z depends on the system which is represented by G, as well as the number of possible states for the objects and the constants associated with the system. The number of allowed colors is hidden within the delta function.

3.2 Bad Colorings

If we think of the states as colors, it becomes clear that the Ising and Potts models are simply graph coloring problems. To make this observation more rigorous, we need to begin with bad colorings, those colorings that are not proper.

Suppose G is a graph with n edges. For each j from 0 to n, define $b_j(G, k)$ to be the number of k-colorings of G with j *bad edges*, those edges that have endpoints which are the same color. Based on this definition, we see that $b_0(G, k)$ is the chromatic polynomial for G.

Since only the bad edges contribute to the energy of the system, it seems likely that we could use these polynomials that count bad edges to rewrite the partition function. If we let j_σ be the number of bad edges in state σ, equation 10 indicates

$$Z(G; Q, \beta J) = \sum_{\sigma} e^{-\beta J(n-j_\sigma)}.$$

Now let us group all states with the same number of bad edges. Note that for each j, the number of such states is $b_j(G; Q)$. By summing over j instead of over each state, we see that

$$Z(G; Q, \beta J) = \sum_{j=0}^{n} b_j(G; Q) e^{-\beta J(n-j)}.$$

Since $e^{\beta Jn}$ is a constant with respect to j, we can factor it out of the sum. This yields

$$Z(G; Q, \beta J) = e^{-\beta Jn} \sum_{j=0}^{n} b_j(G; Q)(e^{\beta} J)^j. \quad (13)$$

The next definition defines a new polynomial that will be used for its relationship with the Tutte polynomial.

Definition 21 *The* ***bad coloring polynomial*** *is defined as*

$$B(G; k, s) = \sum_{j=0}^{|E|} b_j(G; k) s^j. \quad (14)$$

Based on this definition, we then see that

$$Z(G; Q, \beta J) = e^{-Kn} B(G; Q, e^{\beta} J). \quad (15)$$

We now have the partition function written in terms of a function that describes graph colorings, something more similar to what we have already related to the Tutte polynomial.

3.3 The Tutte Polynomial

As stated previously, the connection between the partition function of the Potts model and the Tutte Polynomial is founded in the recipe theorem. Since we have written the partition function in terms of B, an application of the recipe theorem to B will allow us to write the partition function and the probability of being in a given states as evaluations of the Tutte polynomial.

As with the chromatic polynomial we must first define a new polynomial that satisfies $f(G) = 1$ if G has no edges. Note that when G has no edges, the bad coloring polynomial reduces to the chromatic polynomial. Thus $D(G; k, s) = B(G; k, s)/k^{\kappa(G)}$ satisfies the desired condition. The following theorem indicates that the bad coloring polynomial is an evaluation of the Tutte polynomial as desired.

Theorem 8 *For a fixed value, s, the bad coloring polynomial satisfies*

$$B(G; k, s) = k^{\kappa(G)}(s-1)^{r(G)}T\left(G; \frac{s+k-1}{s-1}, s\right). \tag{16}$$

Proof. We will prove that $D(G; k, s)$ satisfies the conditions of the recipe theorem for graphs. The result will then follow.

Condition 1: To demonstrate that D satisfies condition 1 of the recipe theorem for graphs, we will first show that $b_j(G; k) = b_j(G - e; k) - b_j(G \setminus e; k) + b_{j+1}(G \setminus e; k)$ for any edge that is neither a bridge nor a loop.

Consider an edge, e, that is neither a bridge nor a loop. We will count the colorings of G with j bad edges through two cases.

Case 1: We will first count the colorings of G with j bad edges and that assign both endpoints of e the same color. Such colorings are colorings of $G \setminus e$ with exactly $j - 1$ bad edges. Note that for each coloring of G with this property, there is exactly one corresponding coloring of $G \setminus e$ with $j - 1$ bad edges.

Case 2: We will then count the colorings of G that assign the endpoints of e

different colors. These colorings are also colorings of $G - e$ with exactly j bad edges. Unfortunately, $b_j(G - e; k)$ overcounts this type of coloring. This is because it also counts all colorings with j bad edges and the endpoints of e colored the same. Fortunately, there are exactly $b_j(G \setminus e; k)$ such colorings.

Adding all of these colorings together indicates that

$$b_j(G; k) = b_j(G - e; k) - b_j(G \setminus e; k) + b_{j-1}(G \setminus e; k).$$

If we then sum over j, we find that

$$B(G; x, y) = B(G - e; x, y) + (s - 1)B(G \setminus e; x, y).$$

Division by $k^{\kappa}(G)$ yields

$$D(G; x, y) = D(G - e; x, y) + (s - 1)D(G \setminus e; x, y).$$

This demonstrates that D satisfies condition 1 of the recipe theorem for graphs and that $a = 1$ and $b = s - 1$.

Condition 2: Consider a bridge e with endpoints v and w. We will demonstrate that D satisfies condition 2 of the recipe theorem by first proving that

$$b_j(G; k, s) = (k - 1)b_j(G \setminus e; k, s) + b_{j-1}(G \setminus e; k, s).$$

To do so, we will again consider two cases.

Case 1: We will first count the colorings of G with j bad edges in which the endpoints of e are different colors. Consider a coloring of $G \setminus e$ with exactly j bad edges. Let us think of each of these colorings as a function from V to $\mathbb{Z}_k$. Note that for each coloring, x, of $G \setminus e$, there is a coloring of G with $j + 1$ bad edges with e as one of them. Let us change w to a different color to get rid of this problem. There are $k-1$ other colors to choose from. In order to make everything work out, we must also change the color of each vertex in the same connected component of $G-e$. Say we change the color of w to $x(w)+i$. We will then change each other vertex in this set to $x(v_n)+i$. This produces a coloring of G with j bad edges and e not one of them. There are $(k-1)b_j(G\setminus e)$ of these bad edges.

Case 2: We will then count the colorings of G with j bad edges in which the endpoints of e are the same color. For each such coloring, there is exactly one coloring of $G\setminus e$ with $j-1$ bad edges.

Combining these two types of colorings indicates that

$$b_j(G;k,s) = (k-1)b_j(G\setminus e;k,s) + b_{j-1}(G\setminus e;k,s).$$

If we then sum over j we find that

$$B(G;x,y) = (s+k-1)B(G\setminus e;x,y).$$

Division by $k^{\kappa}(G)$ yeilds

$$D(G;x,y) = (s+k-1)D(G\setminus e;x,y).$$

By theorem 6, this is equivalent to condition 2 of the recipe theorem for graphs and indicates that $x_0 = s+k-1$.

Condition 3: Suppose G has a loop, e. Since there is a loop, $B(G;k,s)$ has no s^0 term. In addition, contracting upon e makes each coloring of G have one fewer bad edge. Thus $B(G;k,s) = sB(G\setminus e,k,s)$. Dividing each side by $k^{\kappa(G)}$ yields $D(G;k,s) = sD(G\setminus e,k,s)$. Thus D satisfies condition 3 of the recipe theorem for graphs and $y_0 = s$.

Therefore, D satisfies all of the conditions of the recipe theorem for graphs. Applying this theorem yields

$$D(G; k, s) = (s-1)^{r(G)} T\left(G; \frac{s+k-1}{s-1}, s\right).$$

Equation 16 can then be obtained upon multiplication by $k^{\kappa(G)}$. ■

Now that we have written the bad coloring polynomial as an evaluation of the Tutte polynomial, we can work backwards to write the probability of being in a given state as an evaluation of the Tutte polynomial. Combining equations 11, 15, and 16 yields the following equation:

$$Pr(\sigma) = e^{\beta J|E|} / \left[e^{\beta E(\sigma)} Q^{\kappa(G)} (e^{\beta J} - 1)^{n-\kappa(G)} T\left(G; \frac{e^{\beta J} + Q - 1}{e^{\beta J} - 1}, e^{\beta J}\right)\right]. \tag{17}$$

Although this equation looks far more complicated than the original equation for the probability of a system being in state σ, this equation is easy to work with because the only part of it that depends on the state is the energy of the state, which is easy to calculate. In addition, there are no sums that need to be computed. This means that this long and complex looking equation may be much easier to solve.

3.4 Example

To demonstrate the use of this equation, we will explore an example. For ease of calculation, we will assume that $\beta = J = 1$, $n = 6$, $Q = 3$ and σ is shown below.

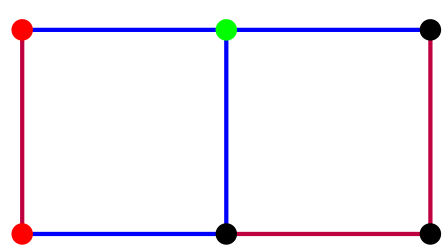

Figure 13: Example for Potts model

Using reduction from the definition, it can be shown that the Tutte polynomial of this graph, G, is given by $T(G;x,y) = x^5 + 2x^4 + 3x^3 + 2x^2 + x + 2x^2y + 2xy + y^2 + y$. In addition, we see that the energy of our state is 4 since $J = 1$. These facts quickly reduce equation 17 to

$$Pr(\sigma) = e^7 / \left[e^3 3(e-1)^{6-1} T\left(G; \frac{e+3-1}{e-1}, e\right)\right].$$

An evaluation of this expression leads to $Pr(\sigma) = .292\%$. From this we can see that is very simple to use this formula when our graph has an easily calculated Tutte polynomial. See [4] for more information on computing the Tutte polynomial.

4 Knot Theory

Our final application of the Tutte polynomial and the recipe theorem comes from the field of knot theory. In this section, we first present some necessary background material on knots and links. Note that this section contains only the knot theory material needed to understand the connection between the HOMFLY polynomial and the Tutte polynomial. For a deeper discussion of knot theory, see [1]. Next, we define the HOMFLY polynomial for all links. We then build links associated to planar graphs. Finally, we use the recipe theorem to write the HOMFLY polynomial of a link in terms of the Tutte polynomial of the related planar graph.

4.1 Knot Theory

Before we can discuss the relationship between links and the Tutte polynomial, we must first understand a bit about knots and links. Knots are very intuitive structures. Take a string. Tie it in a knot in any way you like. Glue the ends together. The result is a knot. We can turn this into a mathematical structure with the following definition.

Definition 22 *A **knot** is a closed curve in space that does not intersect itself.*

Next imagine you repeat the process with more than one string. The result is similar to a knot except that we have multiple strings. This is called a link and is made rigorous in the next definition.

Definition 23 *A **link** is a collection of closed curves in space that do not intersect themselves or one another.*

Note that since a single closed curve can be considered a collection of closed curves, all knots are links. An example of a knot and a link are shown below. In diagrams such as these, the broken strand goes under the connected strand.

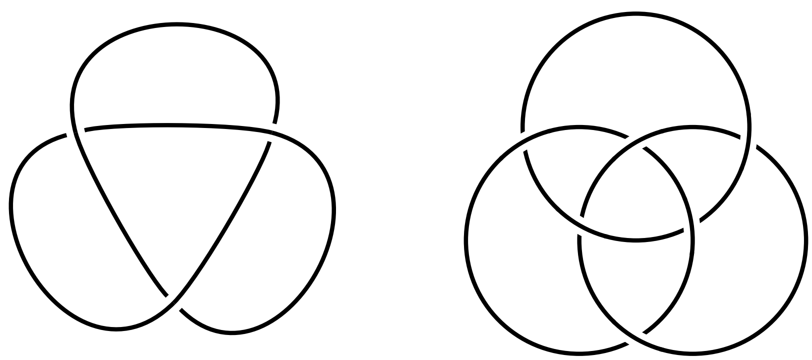

Figure 14: An example knot and link

Since knots are represented by their pictures, called projections, how do we tell if two knots are actually the same? In general, we say that two knots are the same if we can turn one into the other without cutting the string. For example, making the string longer or shorter does not change the knot in the string. Thus, making the curve longer or shorter does not change the knot. In addition, moving the string around away from crossings does not change the knot in the string so neither does this affect the knot in our curve.

Along with these simple moves, there are three more complicated moves, called *Reidemeister moves*, that do not change the knot. As the focus of this text is to understand properties of the Tutte polynomial, we will only discuss the first two.

A *type I Reidemeister move* allows us to twist loops out of knots as shown below.

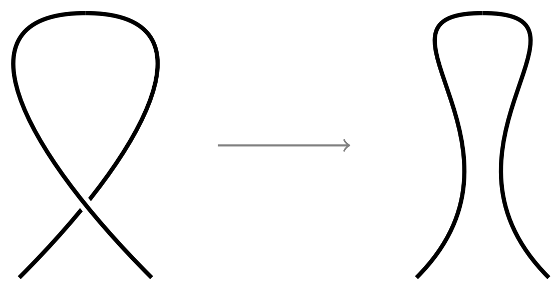

Figure 15: Type I Reidemeister Move

A *type II Reidemeister move* allows us to pull one part of the curve out from under another as shown below.

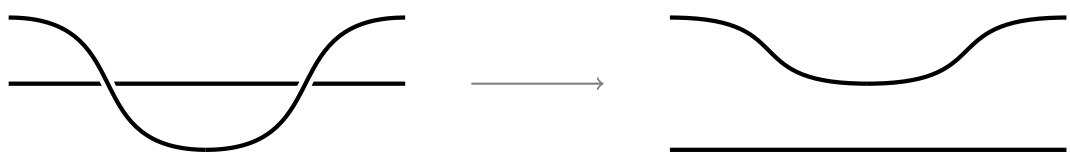

Figure 16: Type II Reidemeister Move

These two moves seem very intuitive because we could do the same things with a string without changing the knot.

In addition, instead of saying two knots are "the same" if we can get from one to the other using Reidemeister moves, we say that two such knots are *isotopic*.

Similar to the concept of a direct sum of matroids as described in section 1, there are two different ways to combine links.

The first is the *connected sum* of L and K and is denoted as $L\#K$. The connected sum of L and K is formed by placing the two links side by side, breaking each of them and connecting them as shown below. Any link that is isotopic to such a knot is considered to be the connected sum of L and K.

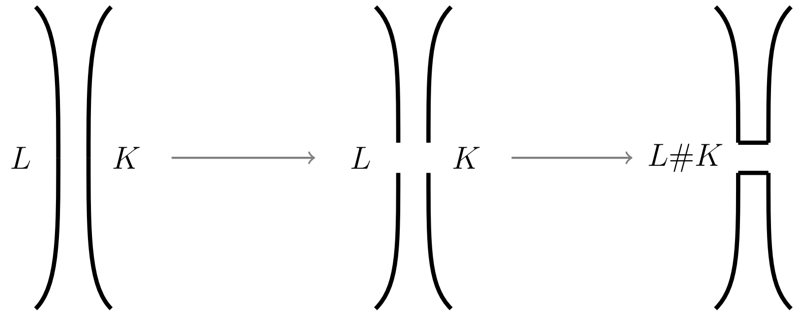

Figure 17: Construction of the connected sum of L and K

The second is simply formed by placing L and K next to one another. We say that J splits into L and K if it is isotopic to this link. Note that figure 14 can be though of as one link, namely the link that splits into the link on the left and the link on the right.

As with the single-vertex graph, there is a knot that is important to understanding the relationship between knots and graph theory.

Definition 24 *The* ***unknot*** *or the* ***trivial knot*** *is the knot that can be drawn without any crossings.*

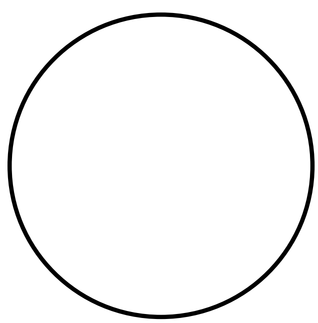

Figure 18: The unknot

With this material in mind, we begin working toward applying the recipe theorem to knot theory.

4.2 The HOMFLY Polynomial

When it comes to actually determining if two knot diagrams are isotopic, using Reidemeister moves is not very efficient. Instead, mathematicians have developed different properties of knots that are unchanged by Reidemeister moves. Such properties are called *invariants*. Because invariants are unchanged by Reidemeister moves, two knots are different if one of their invariants is different. One of these invariants of a link is the HOMFLY polynomial. It is this particular invariant that will be used to relate the Tutte polynomial to properties of links.

Definition 25 *The **HOMFLY polynomial** of a link, L, is a Laurent polynomial in x, y and z that satisfies the following properties:*

1. $P(L, x, y, z) = 0$ *if L is the unknot,*

2. *If L^+, L^- and L° are three diagrams which are identical except inside a small circle where they behave as shown below, then*

$$xP(L^+, x, y, z) + yP(L^-, x, y, z) + zP(L^\circ, x, y, z) = 0, \tag{18}$$

3. *If L and L' are isotopic oriented links then $P(L, x, y, z) = P(L', x, y, z)$.*

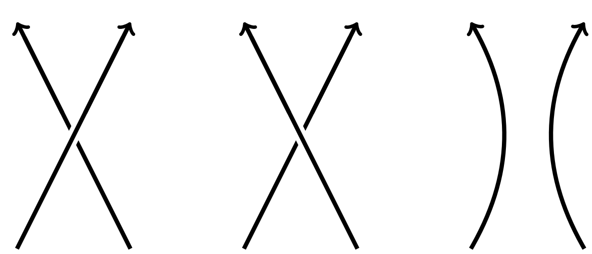

Figure 19: L^+, L^- and L°

Two other interesting properties of the HOMFLY polynomial are listed in the following theorems.

Theorem 9 *If L is split into L_1 and L_2, then*

$$P(L, x, y, z) = \frac{-x - y}{z} P(L_1, x, y, z) P(L_2, x, y, z).$$

Theorem 10 *If L is the connected sum of L_1 and L_2, then*

$$P(L, x, y, z) = P(L_1, x, y, z) P(L_2, x, y, z).$$

Now that the HOMFLY polynomial and a few of its properties have been established, we are ready to relate the Tutte polynomial to the HOMFLY polynomial.

4.3 Building Links from Graphs

In this section, we will begin with a graph and produce a link associated with that graph. For this process to work, we must begin with a *planar graph*, one that can be drawn such that edges only meet at vertices. All graphs discussed in this paper are planar graphs.

Suppose we begin with a planar graph G. We will place a new vertex in the middle of each edge of G. Each of these vertices will have 4 edges coming from it. These edges will be drawn, one on each side of the original edge in each direction. These edges will follow the edges of G until reaching another new vertex. In the case of a vertex of G with no adjacent edges, we draw an edge around it with no incident vertices. Once all of these edges have been drawn, we remove all of the original vertices and edges of G. We are left with the *medial graph*, $M(G)$. A diagram depicting this process is shown below. For another description and example, see [2].

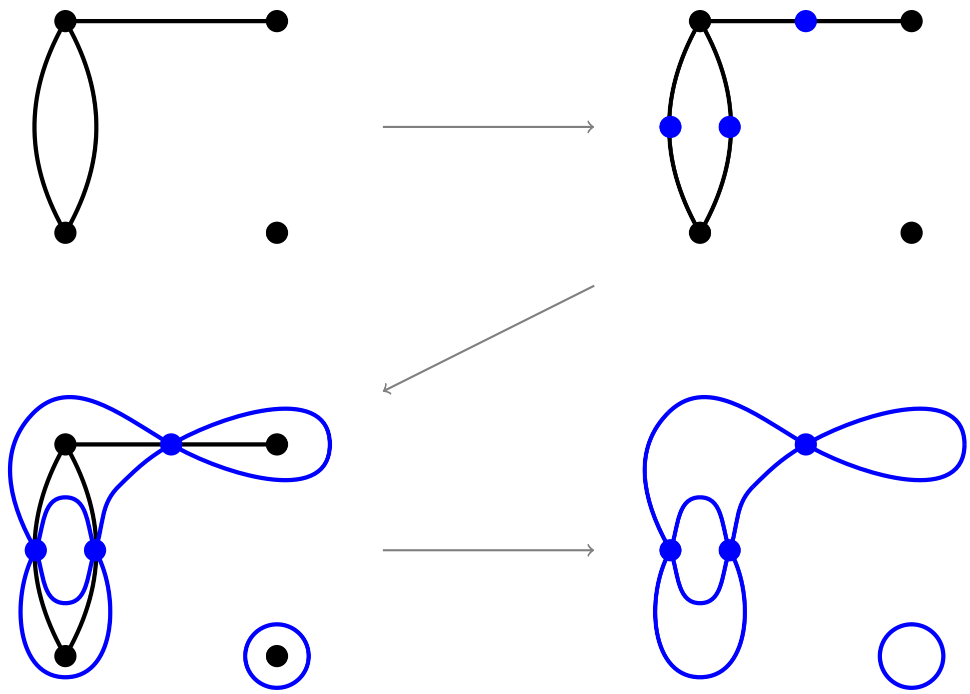

Figure 20: Construction of the medial graph

In order to create a link from the medial graph, we must first assign a direction to each edge. To do this, we color each empty space such that each edge has one shaded region and one white region next to it. For simplicity sake, we color the infinite region on the outside white. Once colored, we assign each edge a direction such that the shaded region is on the left side of the edge. This results in the picture shown below.

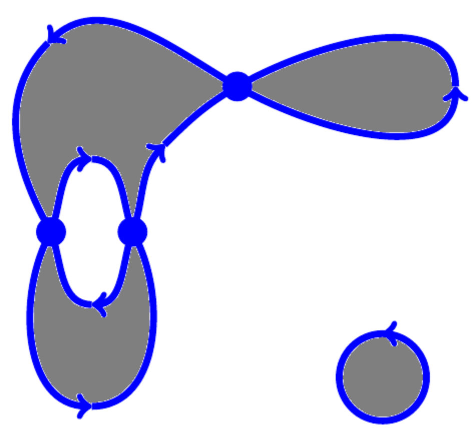

Figure 21: Orientation of the medial graph

It is from this shaded and directed graph that we produce a knot associated with G. To do this, we make the following replacement at each vertex.

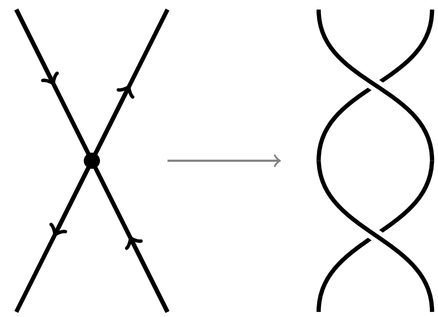

Figure 22: Transition from medial graph to knot

Once this replacement has been made, we have transformed G into an associated link, called $D(G)$, as shown below.

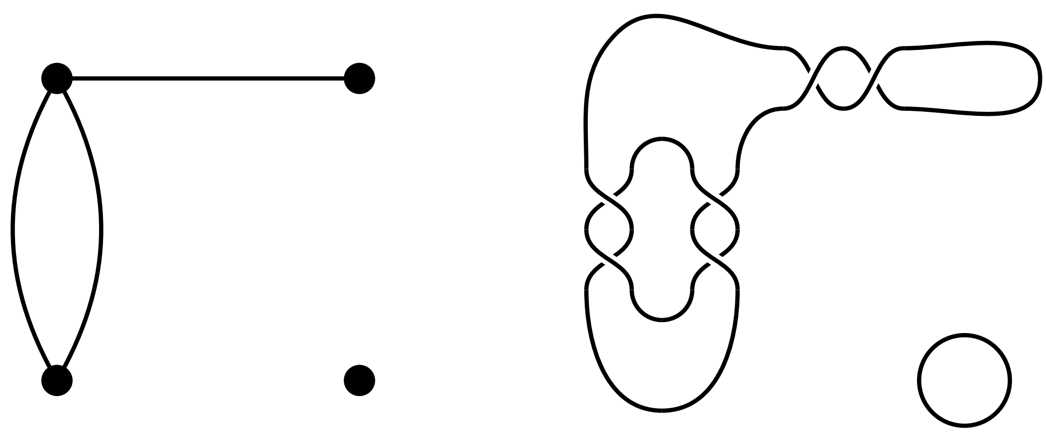

Figure 23: A graph, G, and its associated link, $D(G)$

For another description of this full process, see [3].

4.4 The Tutte Polynomial

The next theorem relates the Tutte polynomial of G to the HOMFLY polynomial of $D(G)$.

Theorem 11 *If G is a connected plane graph, then for all nonzero numbers x, y, z*

$$P(D(G), x, y, z) = \left(\frac{y}{z}\right)^{|V(G)|-1} \left(\frac{-z}{x}\right)^{|E(G)|} T\left(G, \frac{-x}{y}, 1 - \left(\frac{xy + y^2}{z^2}\right)\right). \qquad (19)$$

Proof. We will show that P satisfies the conditions of the recipe theorem for graphs using induction on the number of edges in G. The result will then follow. An similar proof of this theorem that does not use the recipe theorem can be found in [3].

Before working with the conditions of the recipe theorem for graphs, we will manipulate equation 19 to produce an equation true for any edge. It is manipulation of this equation that will yield the conditions of the recipe theorem for graphs.

Consider an edge in G. Using figure 23 as a reference, we can see that figure 24 replaced the edge during the transition to $D(G)$.

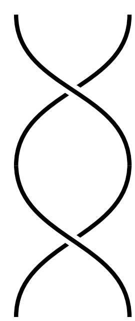

Figure 24: Portion of $D(G)$ corresponding to an edge in G

An application of equation 18 of the HOMFLY polynomial yields the following pictorial equation. In this equation, a picture represents the HOMFLY polynomial of a link which differs from the others only inside the region shown.

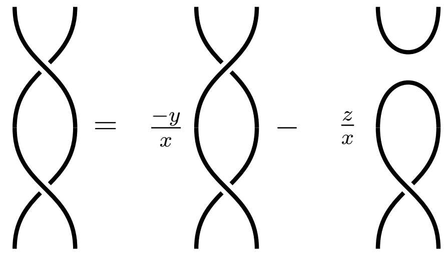

Figure 25: A pictorial representation of property 3 of the HOMFLY polynomial

Applying a type II Reidemeister move to the second and a type I Reidemeister move to the third yields the following simplified pictorial equation.

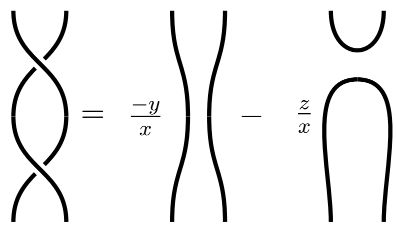

Figure 26: A simplified pictorial equation

From this equation, it is clear that contracting upon e produces the middle picture and deleting e gives the last. Therefore, our equation reduces to

$$P(D(G), x, y, z) = \frac{-y}{x} P(D(G \setminus e), x, y, z) - \frac{z}{x} P(D(G - e), x, y, z).$$

Condition 1: Not that this equation is exactly is condition 1 of the recipe theorem when applied to an edge that is neither a bridge nor a loop. This establishes that $b = -y/x$ and $a = -z/x$.

Condition 2: Next consider an edge, e that is a bridge in G. Since we can see that we can use Reidemeister moves to take us from $D(G)$ to $D(G \setminus e)$, we see that

$$P(D(G), x, y, z) = P(D(G \setminus e), x, y, z).$$

Because of theorem 6, this is equivalent to condition 2 for the recipe theorem. This establishes that $x_0 = 1$.

Condition 3: Finally consider an edge, e that is a loop in G. If we look at our second pictorial equation, we will let $D(G)$ be the link represented by the left picture, L be the link represented by the center picture and L' be the link represented by the right picture. Note that these three links only differ in the region shown. Thus, we see that L' is the connected sum $L_1 \# L_2$ where L_1 is the knot to the left of the region shown and L_2 is the knot to the right of the region shown. Since L is disconnected, we see that L splits into L_1 and L_2. Thus, by theorems 9 and 10, we see that

$$P(L, x, y, z) = \frac{-x + y}{z} P(L', x, y, z).$$

In addition, we see that L' is isotopic to $D(G - e)$. Combining the above equation with the original pictorial equation, we then see that

$$P(D(G), x, y, z) = \left(\frac{xy + y^2}{xz} - \frac{z}{x} \right) P(D(G - e), x, y, z).$$

Because of theorem 6, this is equivalent to condition 3 for the recipe theorem and establishes that $y_0 = \frac{xy+y^2}{xz} - \frac{z}{x}$.

Applying the values for a, b, x_0 and y_0 to the result of the recipe theorem quickly reveals the desired relationship. ■

4.5 Example

To demonstrate how theorem 11 is used, we will find the HOMFLY polynomial of our example graph. Both G and $D(G)$ are shown below for reference.

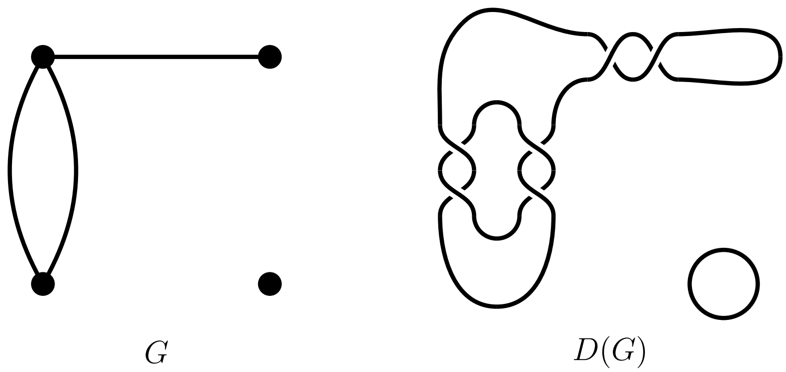

Figure 27: G and $D(G)$

The Tutte polynomial for this graph is quite easy to find and is given by

$$T(G; x, y) = x(x + y).$$

Combining this formula with equation 19 and noting that G has 4 vertices and 3 edges results in the following:

$$P(D(G), x, y, z) = \left(\frac{y}{z}\right)^{4-1} \left(\frac{-z}{x}\right)^{3} \left[\frac{-x}{y}\left(\frac{-x}{y} + 1 - \frac{xy + y^2}{z^2}\right)\right]$$

where the portion in square brackets is the evaluation of the Tutte polynomial. A bit of algebraic simplification then yields

$$P(D(G); x, y, z) = \left(\frac{y^2}{z^3 x^2}\right)\left(1 - \frac{x}{y} - \frac{xy + y^2}{z^2}\right). \tag{20}$$

As with the Potts model, if we begin with a graph whose Tutte polynomial is easily calculated, we can use the relationship between the HOMFLY polynomial and the Tutte polynomial to make calculations very quickly.

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